# Discontinuous Galerkin method for problems in time-dependent domains – theory and applications

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PANM 21, Jablonec nad Nisou, 19–24.6.2022

# Introduction

## Motivation

- Most of the results on the theory and numerical analysis are obtained under the assumption that a **space domain $\Omega$ is independent of time $t$**.
- Problems in **time-dependent domains $\Omega_t$** are important in a number of areas of science and technology.

## The main goal

To work out an **accurate, efficient and robust numerical method** for the solution of problems in time-dependent domains.

# Why discontinuous Galerkin method?

## FEM - finite element method

- piecewise polynomial continuous approximation
- high order of accuracy
- suitable for problems with continuous solutions

## FVM - finite volume method

- piecewise constant discontinuous approximation
- low order of accuracy
- suitable for problems with discontinuities or steep gradients

## DGM - discontinuous Galerkin method

- piecewise polynomial discontinuous approximation
- high order of accuracy
- efficient for problems with continuous solutions as well as discontinuities or steep gradients

# Overview


1. Formulation of a continuous model problem and its discretization
   - Arbitrary Lagrangian-Eulerian (ALE) method
   - ALE space-time discretization
   - ALE-STDG approximate solution
2. Stability analysis
   - Estimates of individual terms
   - Discrete characteristic function
   - Main theorem
3. Error estimates
   - Abstract error estimate
   - Error estimate in terms of $h$ and $\tau$
4. Applications of the STDGM
   - Nonlinear elasticity benchmark problem
   - Flow induced vocal folds vibrations
5. Conclusion

# Formulation of the continuous problem

## Nonlinear convection-diffusion problem in a time-dependent domain

Find $u = u(x,t)$ with $x \in \Omega_t \subset \mathbb{R}^d$, $t \in (0,T)$ such that

$$\frac{\partial u}{\partial t} + \sum_{s=1}^{d} \frac{\partial f_s(u)}{\partial x_s} - \operatorname{div}(\beta(u)\nabla u) = g \qquad \text{in } \Omega_t,\ t \in (0,T), \tag{1}$$

$$u = u_D \qquad \text{on } \partial\Omega_t,\ t \in (0,T), \tag{2}$$

$$u(x,0) = u^0(x), \qquad x \in \Omega_0. \tag{3}$$

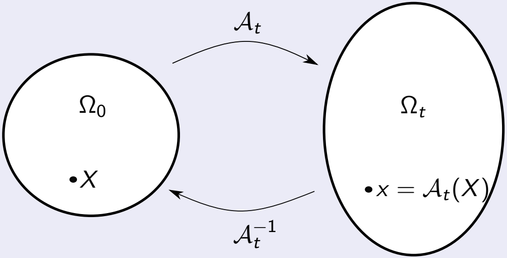
Assume that

- $f_s \in C^1(\mathbb{R})$, $f_s(0) = 0$ and $|f_s'| \le L_f$, $\quad s = 1,\dots,d$,
- function $\beta$ is bounded and Lipschitz-continuous

$$\beta : \mathbb{R} \to [\beta_0, \beta_1], \quad 0 < \beta_0 < \beta_1 < \infty,$$
$$|\beta(u_1) - \beta(u_2)| \le L_\beta |u_1 - u_2| \quad \forall u_1, u_2 \in \mathbb{R}.$$

# ALE method

## ALE mapping




- one-to-one mapping of the reference domain to the current configuration
- **domain velocity**: $\tilde{\boldsymbol{z}}(X,t) = \frac{\partial}{\partial t}\mathcal{A}_t(X), \quad \boldsymbol{z}(x,t) = \tilde{\boldsymbol{z}}(\mathcal{A}_t^{-1}(x),t),$
- **ALE derivative**: $D_t f(x,t) = \frac{D}{Dt} f(x,t) = \frac{\partial}{\partial t} f(\mathcal{A}_t(X),t),$
- The use of the **chain rule** yields: $\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \boldsymbol{z} \cdot \nabla f$

# Reformulation of the problem

## Problem (1)–(3) in the ALE form:

Find $u = u(x,t)$ with $x \in \Omega_t$, $t \in (0,T)$ such that

$$\frac{Du}{Dt} + \sum_{s=1}^{d} \frac{\partial f_s(u)}{\partial x_s} - \boldsymbol{z} \cdot \nabla u - \mathrm{div}(\beta(u)\nabla u) = g \qquad \text{in } \Omega_t, \tag{4}$$

$$u = u_D \qquad \text{on } \partial\Omega_t, \tag{5}$$

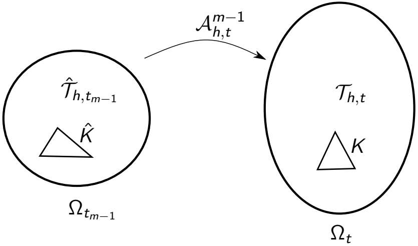
$$u(x,0) = u^0(x), \qquad x \in \Omega_0. \tag{6}$$

## Numerical method

For the numerical solution we use the **ALE space-time discontinuous Galerkin** (ALE-STDG) discretization.

# ALE space-time discretization

- Partition of the time interval $[0, T]$: $0 = t_0 < t_1 < \cdots < t_M = T$
  $I_m = (t_{m-1}, t_m)$, $\tau_m = t_m - t_{m-1}$
- advantage of the STDGM: on every time interval $\bar{I}_m = [t_{m-1}, t_m]$ it is possible to consider a different space partition
- ALE mapping separately on each time interval $[t_{m-1}, t_m)$:
  $\mathcal{A}_{h,t}^{m-1} : \overline{\Omega}_{t_{m-1}} \xrightarrow{\text{onto}} \overline{\Omega}_t \quad$ for $t \in [t_{m-1}, t_m)$




- $\mathcal{A}_{h,t}^{m-1}$ is in space a piecewise affine mapping on $\hat{\mathcal{T}}_{h,t_{m-1}}$, continuous in space variable $X \in \Omega_{t_{m-1}}$ and continuously differentiable in time $t$

# Discrete function spaces

## Piecewise polynomial functions in space

$p \geq 1$, $P^p(\hat{K})$ is the space of all polynomials on $\hat{K}$ of degree $\leq p$

$$S_h^{p,m-1} = \left\{ \varphi \in L^2(\Omega_{t_{m-1}});\ \varphi|_{\hat{K}} \in P^p(\hat{K}) \quad \forall\, \hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}} \right\}$$

## Piecewise polynomial functions in space and time

$p,\ q \geq 1$

$$\begin{aligned} S_{h,\tau}^{p,q} \quad &= \quad \Big\{ \varphi;\ \varphi\left( \mathcal{A}_{h,t}^{m-1}(X), t \right) = \sum_{i=0}^{q} \vartheta_i(X)\, t^i, \\ &\qquad \vartheta_i \in S_h^{p,m-1},\ X \in \Omega_{t_{m-1}},\ t \in \overline{I}_m,\ m = 1, \dots, M \Big\} \end{aligned}$$

# Notation

## Notation of faces and elements:

- $\mathcal{F}_{h,t}$ - system of all faces of all elements $K \in \mathcal{T}_{h,t}$:

$$\mathcal{F}_{h,t} = \mathcal{F}_{h,t}^{I} \cup \mathcal{F}_{h,t}^{B}$$

- $K_\Gamma^{(L)}$, $K_\Gamma^{(R)} \in \mathcal{T}_{h,t}$ - adjacent elements to the face $\Gamma \in \mathcal{F}_{h,t}^{I}$
- for $\Gamma \in \mathcal{F}_{h,t}^{B}$ the element adjacent to this face will be denoted by $K_\Gamma^{(L)}$
- $\mathbf{n}_\Gamma$ is the outer normal to $\partial K_\Gamma^{(L)}$

## Broken Sobolev space

- over a triangulation $\mathcal{T}_{h,t}$, for each positive integer $k$, we define the space

$$H^k(\Omega_t, \mathcal{T}_{h,t}) = \{v;\, v|_K \in H^k(K) \quad \forall K \in \mathcal{T}_{h,t}\},$$

# Notation

## Average and jump of traces of $v \in H^1(\Omega_t, \mathcal{T}_{h,t})$

- $v_\Gamma^{(L)}$, $v_\Gamma^{(R)}$ - traces of $v$ on $\Gamma \in \mathcal{F}_{h,t}$ from the side of elements $K_\Gamma^{(L)}$ and $K_\Gamma^{(R)}$, respectively
- $\langle v \rangle_\Gamma = \frac{1}{2}\left(v_\Gamma^{(L)} + v_\Gamma^{(R)}\right)$ - average of traces of $v$ on $\Gamma \in \mathcal{F}_{h,t}^I$
- $[v]_\Gamma = v_\Gamma^{(L)} - v_\Gamma^{(R)}$ - jump of traces of $v$ on $\Gamma \in \mathcal{F}_{h,t}^I$

## Diameter of an element and face

- $h_K = \operatorname{diam} K$ for $K \in \mathcal{T}_{h,t}$
- $h(\Gamma) = \operatorname{diam} \Gamma$ for $\Gamma \in \mathcal{F}_{h,t}$

## Jump over a time interval

For a function $\varphi$ defined in $\bigcup_{m=1}^{M} I_m$ we denote

$$\varphi_m^\pm = \varphi(t_m\pm) = \lim_{t \to t_m\pm} \varphi(t), \quad \{\varphi\}_m = \varphi(t_m+) - \varphi(t_m-)$$

# ALE-STDG approximate solution

- choose an arbitrary but fixed $t \in I_m$, multiply equation (4) by a test function $\varphi$, integrate over an element $K$, sum over all elements $K \in \mathcal{T}_{h,t}$
- apply Green's theorem to the convection and diffusion terms, introduce the concept of a numerical flux and suitable expressions mutually vanishing

## Definition

A function $U$ is an approximate solution of problem (4)–(6), if $U \in S_{h,\tau}^{p,q}$ and

$$\int_{I_m} \left( (D_t U, \varphi)_{\Omega_t} + a_h(U, \varphi, t) + \beta_0 J_h(U, \varphi, t) + b_h(U, \varphi, t) + d_h(U, \varphi, t) \right) \mathrm{d}t$$

$$+(\{U\}_{m-1}, \varphi_{m-1}^{+})_{\Omega_{t_{m-1}}} = \int_{I_m} l_h(\varphi, t) \, \mathrm{d}t \quad \forall \varphi \in S_{h,\tau}^{p,q}, \quad m = 1, \ldots, M,$$

$$U_0^- \in S_h^{p,0}, \quad (U_0^- - u^0, v_h)_{\Omega_0} = 0 \quad \forall v_h \in S_h^{p,0}.$$

# ALE-STDG approximate solution

## Definition

A function $U \in S^{p,q}_{h,\tau}$ is an approximate solution of problem (4)–(6), if

$$\int_{I_m} \left( (D_t U, \varphi)_{\Omega_t} + a_h(U,\varphi,t) + \beta_0 J_h(U,\varphi,t) + b_h(U,\varphi,t) + d_h(U,\varphi,t) \right) \mathrm{d}t$$

$$+(\{U\}_{m-1}, \varphi^+_{m-1})_{\Omega_{t_{m-1}}} = \int_{I_m} l_h(\varphi,t)\,\mathrm{d}t \quad \forall \varphi \in S^{p,q}_{h,\tau}, \quad m = 1,\dots,M,$$

$$a_h(u,\varphi,t) := \sum_{K\in\mathcal{T}_{h,t}} \int_K \beta(u)\nabla u \cdot \nabla\varphi \,\mathrm{d}x$$

$$- \sum_{\Gamma\in\mathcal{F}^I_{h,t}} \int_\Gamma \left( \langle \beta(u)\nabla u\rangle \cdot \boldsymbol{n}_\Gamma \,[\varphi] + \Theta \,\langle \beta(u)\nabla\varphi\rangle \cdot \boldsymbol{n}_\Gamma \,[u] \right) \mathrm{d}S$$

$$- \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma \left( \beta(u)\nabla u \cdot \boldsymbol{n}_\Gamma \,\varphi + \Theta\beta(u)\nabla\varphi \cdot \boldsymbol{n}_\Gamma \,u - \Theta\beta(u)\nabla\varphi\cdot \boldsymbol{n}_\Gamma \,u_D \right) \mathrm{d}S,$$

# ALE-STDG approximate solution

## Definition

A function $U \in S_{h,\tau}^{p,q}$ is an approximate solution of problem (4)–(6), if

$$\int_{I_m} \left( (D_t U, \varphi)_{\Omega_t} + a_h(U,\varphi,t) + \beta_0 J_h(U,\varphi,t) + b_h(U,\varphi,t) + d_h(U,\varphi,t) \right) \mathrm{d}t$$

$$+(\{U\}_{m-1}, \varphi_{m-1}^{+})_{\Omega_{t_{m-1}}} = \int_{I_m} l_h(\varphi,t)\,\mathrm{d}t \quad \forall \varphi \in S_{h,\tau}^{p,q}, \quad m = 1,\ldots,M,$$

$$J_h(u,\varphi,t) := c_W \sum_{\Gamma \in \mathcal{F}_{h,t}^{I}} h(\Gamma)^{-1} \int_\Gamma [u]\,[\varphi]\,\mathrm{d}S + c_W \sum_{\Gamma \in \mathcal{F}_{h,t}^{B}} h(\Gamma)^{-1} \int_\Gamma u\,\varphi\,\mathrm{d}S,$$

$$b_h(u,\varphi,t) := -\sum_{K \in \mathcal{T}_{h,t}} \int_K \sum_{s=1}^{d} f_s(u) \frac{\partial \varphi}{\partial x_s}\,\mathrm{d}x$$

$$+ \sum_{\Gamma \in \mathcal{F}_{h,t}^{I}} \int_\Gamma H(u_\Gamma^{(L)}, u_\Gamma^{(R)}, \boldsymbol{n}_\Gamma)\,[\varphi]\,\mathrm{d}S + \sum_{\Gamma \in \mathcal{F}_{h,t}^{B}} \int_\Gamma H(u_\Gamma^{(L)}, u_\Gamma^{(L)}, \boldsymbol{n}_\Gamma)\,\varphi\,\mathrm{d}S,$$

# ALE-STDG approximate solution

## Definition

A function $U \in S_{h,\tau}^{p,q}$ is an approximate solution of problem (4)–(6), if

$$\int_{I_m} \left( (D_t U, \varphi)_{\Omega_t} + a_h(U,\varphi,t) + \beta_0 J_h(U,\varphi,t) + b_h(U,\varphi,t) + d_h(U,\varphi,t) \right) \mathrm{d}t$$
$$+ (\{U\}_{m-1}, \varphi_{m-1}^{+})_{\Omega_{t_{m-1}}} = \int_{I_m} l_h(\varphi,t)\,\mathrm{d}t \quad \forall \varphi \in S_{h,\tau}^{p,q}, \quad m = 1,\dots,M,$$

$$d_h(u,\varphi,t) := -\sum_{K\in\mathcal{T}_{h,t}} \int_K (\boldsymbol{z}\cdot\nabla u)\varphi\,\mathrm{d}x,$$

$$l_h(\varphi,t) := \sum_{K\in\mathcal{T}_{h,t}} \int_K g\varphi\,\mathrm{d}x + \beta_0\, c_W \sum_{\Gamma\in\mathcal{F}_{h,t}^B} h(\Gamma)^{-1} \int_\Gamma u_D\,\varphi\,\mathrm{d}S$$

# Some auxiliary results

- In the space $H^1(\Omega_t, \mathcal{T}_{h,t})$ and over $\partial\Omega$ we define the norms

$$\|\varphi\|_{DG,t}^2 = \sum_{K \in \mathcal{T}_{h,t}} |\varphi|_{H^1(K)}^2 + J_h(\varphi, \varphi, t)$$

$$\|u_D\|_{DGB,t}^2 = c_W \sum_{\Gamma \in \mathcal{F}_{h,t}^B} h(\Gamma)^{-1} \int_\Gamma |u_D|^2 \, \mathrm{d}S = J_h^B(u_D, u_D, t)$$

- we assume that

$$\mathcal{A}_{h,t}^{m-1} \in W^{1,\infty}(I_m; W^{1,\infty}(\Omega_{t_{m-1}})), \quad m = 1, \dots, M,$$

$$(A_{h,t}^{m-1})^{-1} \in W^{1,\infty}(I_m; W^{1,\infty}(\Omega_t)), \quad m = 1, \dots M$$

- we have $J \in W^{1,\infty}(I_m; L^\infty(\Omega_{t_{m-1}}))$, $J^{-1} \in W^{1,\infty}(I_m; L^\infty(\Omega_t))$
- there exist constants $C_J^-$, $C_J^+ > 0$ such that the Jacobians satisfy

$$C_J^- \le J(X, t) \le C_J^+, \quad X \in \overline{\Omega}_{t_{m-1}}, \ t \in \overline{I}_m, \ m = 1, \dots, M,$$
$$(C_J^+)^{-1} \le J^{-1}(x, t) \le (C_J^-)^{-1}, \quad x \in \overline{\Omega}_t, \ t \in \overline{I}_m, \ m = 1, \dots, M$$

# Basic equation

- we use $\varphi := U$ as a test function and get the basic identity

$$\int_{I_m} \left( (D_t U, U)_{\Omega_t} + a_h(U, U, t) + \beta_0 J_h(U, U, t) + b_h(U, U, t) \right) \mathrm{d}t$$
$$+ \int_{I_m} d_h(U, U, t) \, \mathrm{d}t + (\{U\}_{m-1}, U_{m-1}^{+})_{\Omega_{t_{m-1}}} = \int_{I_m} l_h(U, t) \, \mathrm{d}t$$

- estimates of the individual terms are based on the multiplicative trace inequality, inverse inequality, Young's inequality and assumptions of function $\beta$

# Coercivity of the diffusion and penalty term

## Lemma

*Let*

$$c_W \geq \frac{\beta_1^2}{\beta_0^2} c_M (c_I + 1) \quad \textit{for} \quad \Theta = -1 \ (\textit{NIPG}),$$

$$c_W \geq \frac{\beta_1^2}{\beta_0^2} c_M (c_I + 1) \quad \textit{for} \quad \Theta = 0 \ (\textit{IIPG}),$$

$$c_W \geq \frac{16\beta_1^2}{\beta_0^2} c_M (c_I + 1) \quad \textit{for} \quad \Theta = 1 \ (\textit{SIPG}).$$

*Then*

$$\int_{I_m} \left( a_h(U, U, t) + \beta_0 \, J_h(U, U, t) \right) \mathrm{d}t \geq \frac{\beta_0}{2} \int_{I_m} \|U\|_{DG,t}^2 \, \mathrm{d}t - \frac{\beta_0}{2} \int_{I_m} \|u_D\|_{DGB,t}^2 \, \mathrm{d}t.$$

# Estimates of other terms

**Lemma**

*For each* $k_1, k_2, k_3 > 0$ *there exists a constant* $c_b, c_d > 0$ *such that for the approximate solution* $U$ *of problem (4)–(6) we have the inequalities*

$$\int_{I_m} |b_h(U,U,t)| \mathrm{d}t \le \frac{\beta_0}{k_1} \int_{I_m} \|U\|^2_{DG,t} \mathrm{d}t + c_b \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t,$$

$$\int_{I_m} |d_h(U,U,t)| \, \mathrm{d}t \le \frac{\beta_0}{2k_2} \int_{I_m} \|U\|^2_{DG,t} \, \mathrm{d}t + \frac{c_d}{2\beta_0} \int_{I_m} \|U\|^2_{\Omega_t} \, \mathrm{d}t,$$

$$\int_{I_m} |l_h(U,t)| \, \mathrm{d}t \le \frac{1}{2} \int_{I_m} \left( \|g\|^2_{\Omega_t} + \|U\|^2_{\Omega_t} \right) \, \mathrm{d}t$$

$$+\beta_0 k_3 \int_{I_m} \|u_D\|^2_{DGB,t} \, \mathrm{d}t + \frac{\beta_0}{k_3} \int_{I_m} \|U\|^2_{DG,t} \, \mathrm{d}t.$$

# Estimate of the term with the ALE derivative

**Lemma**

*It holds that*

$$\int_{I_m} (D_t U, U)_{\Omega_t} \,\mathrm{d}t \geq \frac{1}{2}\left( \|U_m^-\|^2_{\Omega_{t_m}} - \|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} - c_z \int_{I_m} \|U\|^2_{\Omega_t} \,\mathrm{d}t \right),$$

$$\left(\{U\}_{m-1}, U_{m-1}^+\right)_{\Omega_{t_{m-1}}} = \frac{1}{2}\left( \|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} + \|\{U\}_{m-1}\|^2_{\Omega_{t_{m-1}}} - \|U_{m-1}^-\|^2_{\Omega_{t_{m-1}}} \right),$$

$$\int_{I_m} (D_t U, U)_{\Omega_t} \,\mathrm{d}t + \left(\{U\}_{m-1}, U_{m-1}^+\right)_{\Omega_{t_{m-1}}}$$

$$\geq \frac{1}{2}\|U_m^-\|^2_{\Omega_{t_m}} + \frac{1}{2}\|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} - \frac{c_z}{2} \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t - \left(U_{m-1}^-, U_{m-1}^+\right)_{\Omega_{t_{m-1}}}.$$

# Estimates of the basic idientity

$$\underbrace{\int_{I_m} (D_t U, U)_{\Omega_t} \mathrm{d}t}_{\geq \frac{1}{2}\|U_m^-\|^2_{\Omega_{t_m}} - \frac{1}{2}\|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} - \frac{c_z}{2}\int_{I_m}\|U\|^2_{\Omega_t}\,\mathrm{d}t} + \underbrace{\int_{I_m} a_h(U,U,t) + \beta_0 J_h(U,U,t)\,\mathrm{d}t}_{\geq \frac{\beta_0}{2}\int_{I_m}\|U\|^2_{DG,t}\mathrm{d}t - \frac{\beta_0}{2}\int_{I_m}\|u_D\|_{DGB,t}\,\mathrm{d}t}$$

$$+ \underbrace{\int_{I_m} b_h(U,U,t)\,\mathrm{d}t}_{\leq \frac{\beta_0}{2k_1}\int_{I_m}\|U\|^2_{DG,t}\mathrm{d}t + c_b\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t} + \underbrace{\int_{I_m} d_h(U,U,t)\,\mathrm{d}t}_{\leq \frac{\beta_0}{2k_2}\int_{I_m}\|U\|^2_{DG,t}\mathrm{d}t + \frac{c_d}{2\beta_0}\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t}$$

$$+ \underbrace{\left(\{U\}_{m-1}, U_{m-1}^+\right)_{\Omega_{t_{m-1}}}}_{= \frac{1}{2}\left(\|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} + \|\{U\}_{m-1}\|^2_{\Omega_{t_{m-1}}} - \|U_{m-1}^-\|^2_{\Omega_{t_{m-1}}}\right)}$$

$$= \underbrace{\int_{I_m} l_h(U,t)\,\mathrm{d}t}_{\leq \frac{1}{2}\int_{I_m}\left(\|g\|^2_{\Omega_t} + \|U\|^2_{\Omega_t}\right)\mathrm{d}t + \frac{\beta_0 k_3}{2}\int_{I_m}\|u_D\|^2_{DGB,t}\mathrm{d}t + \frac{\beta_0}{2k_3}\int_{I_m}\|U\|^2_{DG,t}\mathrm{d}t}$$

# Estimates of the basic idientity

- after multiplying by two, rearranging, choosing $k_1 = k_2 = k_3 = 6$, setting $C_{T2} = \max\{1, 7\beta_0, c_z + 1 + c_d/\beta_0 + 2c_b\}$ we get

$$\|U_m^-\|^2_{\Omega_{t_m}} - \|U_{m-1}^-\|^2_{\Omega_{t_{m-1}}} + \|\{U\}_{m-1}\|^2_{\Omega_{t_{m-1}}} + \frac{\beta_0}{2}\int_{I_m} \|U\|^2_{DG,t}\mathrm{d}t$$
$$\leq C_{T2}\left(\int_{I_m} \|g\|^2_{\Omega_t}\mathrm{d}t + \int_{I_m} \|u_D\|^2_{DGB,t}\mathrm{d}t + \int_{I_m} \|U\|^2_{\Omega_t}\mathrm{d}t\right)$$

- we need to estimate $\int_{I_m} \|U\|^2_{\Omega_t}$ in terms of data $g, u_D, U_0$

# Discrete characteristic function

- approximate solution in $\Omega_t$: $U = U(x,t), \quad x \in \Omega_t,\ t \in I_m$
- approximate solution transformed to the reference domain $\Omega_{t_{m-1}}$:
  $\tilde{U} = \tilde{U}(X,t) = U(\mathcal{A}_t(X),t), \quad X \in \Omega_{t_{m-1}},\ t \in I_m$
- For $s \in I_m$ by $\tilde{\mathcal{U}}_s = \tilde{\mathcal{U}}_s(X,t)$, X$\in \Omega_{t_{m-1}}$, $t \in I_m$, we denote the discrete characteristic function to $\tilde{U}$ at a point $s \in I_m$.
  It is defined as $\tilde{\mathcal{U}}_s \in P^q(I_m; S_h^{p,m-1})$ such that

$$
\begin{aligned}
\int_{I_m} (\tilde{\mathcal{U}}_s, \varphi)_{\Omega_{t_{m-1}}} \,\mathrm{d}t &= \int_{t_{m-1}}^{s} (\tilde{U}, \varphi)_{\Omega_{t_{m-1}}} \,\mathrm{d}t \quad \forall \varphi \in P^{q-1}(I_m; S_h^{p,m-1}), \\
\tilde{\mathcal{U}}_s(X, t_{m-1}+) &= \tilde{U}(X, t_{m-1}+),\ X \in \Omega_{t_{m-1}}.
\end{aligned}
$$

- by $\mathcal{U}_s = \mathcal{U}_s(x,t)$, $x \in \Omega_t, t \in I_m$ we denote the discrete characteristic function to $U \in S_{h,\tau}^{p,q}$ at a point $s \in I_m$:

$$
\mathcal{U}_s(x,t) = \tilde{\mathcal{U}}_s(\mathcal{A}_t^{-1}(x),t),\ x \in \Omega_t,\ \ t \in I_m
$$

- hence, $\mathcal{U}_s \in S_{h,\tau}^{p,q}$ and for $X \in \Omega_{t_{m-1}}$ we have

$$
\mathcal{U}_s(X, t_{m-1}+) = U(X, t_{m-1}+)
$$

# Continuity of the discrete characteristic function

- the discrete characteristic function mapping $U \to \mathcal{U}_s$ is continuous with respect to the norms $\|\cdot\|_{L^2(\Omega_t)}$ and $\|\cdot\|_{DG,t}$
- there exists a constant $\tilde{c}_{CH}^{(1)} > 0$ depending on $q$ only such that

$$\int_{I_m} \|\tilde{\mathcal{U}}_s\|^2_{\Omega_{t_{m-1}}}\,\mathrm{d}t \;\le\; \tilde{c}_{CH}^{(1)} \int_{I_m} \|\tilde{U}\|^2_{\Omega_{t_{m-1}}}\,\mathrm{d}t,$$

**Theorem**

*There exist constants $c_{CH}^{(1)}, c_{CH}^{(2)} > 0$, such that*

$$\int_{I_m} \|\mathcal{U}_s\|^2_{\Omega_t}\,\mathrm{d}t \;\le\; c_{CH}^{(1)} \int_{I_m} \|U\|^2_{\Omega_t}\,\mathrm{d}t$$

$$\int_{I_m} \|\mathcal{U}_s\|^2_{DG,t}\,\mathrm{d}t \;\le\; c_{CH}^{(2)} \int_{I_m} \|U\|^2_{DG,t}\,\mathrm{d}t$$

*for all $s \in I_m$, $m = 1, \ldots, M$ and $h \in (0, \overline{h})$.*

# Estimate of $\int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t$

- $t_{m-1+l/q} = t_{m-1} + \tau_m \frac{l}{q}, \quad l = 0, \dots, q$
- denote $\mathcal{U}_l^*$ and $\tilde{\mathcal{U}}_l^*$ the discrete characteristic functions to $U$ and $\tilde{U}$, respectively at the time instant $t_{m-1+l/q}$, $l = 0, \dots, q$
- estimate for $\int_{I_m} (D_t U, \mathcal{U}_s)_{\Omega_t} \, \mathrm{d}t + (\{U\}_{m-1}, \mathcal{U}_s(t^+_{m-1}))_{\Omega_{t_{m-1}}}$
- estimates from above for $|a_h(U, \mathcal{U}_l^*, t) + \beta_0 J_h(U, \mathcal{U}_l^*, t)|$, $\int_{I_m} |b_h(U, \mathcal{U}_l^*, t)| \mathrm{d}t$, $\int_{I_m} |d_h(U, \mathcal{U}_l^*, t)| \, \mathrm{d}t$, $\int_{I_m} |l_h(\mathcal{U}_l^*, t)| \, \mathrm{d}t$

## Theorem

*There exist constants $C_{T4}, C^*_{T4} > 0$ such that*

$$\int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t \leq C_{T4} \tau_m \left( \|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}} + \int_{I_m} \left( \|g\|^2_{\Omega_t} + \|u_D\|^2_{DGB,t} \right) \mathrm{d}t \right)$$

*provided* $0 < \tau_m < C^*_{T4}$.

# Unconditional stablility - main theorem

**Theorem**

*Let* $0 < \tau_m \le C^*_{T4}$ *for* $m = 1, \dots, M$. *Then there exists a constant* $C_{T5} > 0$ *such that*

$$\|U^-_m\|^2_{\Omega_{t_m}} + \sum_{j=1}^m \|\{U\}_{j-1}\|^2_{\Omega_{t_{j-1}}} + \frac{\beta_0}{2} \sum_{j=1}^m \int_{I_j} \|U\|^2_{DG,t}\, \mathrm{d}t$$

$$\le C_{T5} \left( \|U^-_0\|^2_{\Omega_{t_0}} + \sum_{j=1}^m \int_{I_j} R_{t,j}\, \mathrm{d}t \right), \quad m = 1, \dots, M,\ h \in (0, \overline{h}),$$

*where* $R_{t,j} = (C_{T2} + C_{T4}\,\tau_j)\,(\|g\|^2_{\Omega_t} + \|u_D\|^2_{DGB,t})$ *for* $t \in I_j$.

- proof is based on the discrete Gronwall inequality
- constant $C_{T5} := \exp(C_{T2} C_{T4} T)$ is independent of the time step $\tau_m$

# Error estimates

- estimation of the error $e = U - u$

$$e = \xi + \eta, \quad \text{where } \xi = U - \pi u \in S_{h,\tau}^{p,q} \quad \text{and } \eta = \pi u - u.$$

- $\pi$ is a projection into the space $S_{h,\tau}^{p,q}$
- subtracting identities for $U$ and $u$ and setting $\varphi := \xi$ we get

$$\begin{aligned}
&\int_{I_m} \left( (D_t\xi, \xi)_{\Omega_t} + a_h(U, U, \xi, t) - a_h(U, \pi u, \xi, t) \right) \, \mathrm{d}t \\
&\quad + \int_{I_m} \left( \beta_0 J_h(\xi, \xi, t) + d_h(\xi, \xi, t) \right) \, \mathrm{d}t + \left( \{\xi\}_{m-1}, \xi_{m-1}^{+} \right)_{\Omega_{t_{m-1}}} \\
&= \int_{I_m} \left( -a_h(U, \pi u, \xi, t) + a_h(u, \pi u, \xi, t) - a_h(u, \pi u, \xi, t) + a_h(u, u, \xi, t) \right) \, \mathrm{d}t \\
&\quad + \int_{I_m} \left( b_h(u, \xi, t) - b_h(U, \xi, t) - \beta_0 J_h(\eta, \xi, t) - d_h(\eta, \xi, t) \right) \, \mathrm{d}t \\
&\quad - \int_{I_m} (D_t\eta, \xi)_{\Omega_t} \, \mathrm{d}t - \left( \{\eta\}_{m-1}, \xi_{m-1}^{+} \right)_{\Omega_{t_{m-1}}} .
\end{aligned}$$

# Abstract error estimate

**Theorem**

*Let $0 < \tau_m \leq C^*_{T9}$ for $m = 1, \dots, M$. Then there exists a constant $C_{AE} > 0$ such that*

$$\|e_m^-\|^2_{\Omega_{t_m}} + \frac{1}{4}\sum_{j=1}^{m} \|\{e\}_{j-1}\|^2_{\Omega_{t_{j-1}}} + \frac{\beta_0}{2}\sum_{j=1}^{m}\int_{I_j} \|e\|^2_{DG,t}\,\mathrm{d}t$$

$$\leq C_{AE}\Big( \|\xi_0^-\|^2_{\Omega_{t_0}} + \sum_{j=1}^{m}\int_{I_j} C^* K_{t,j}(\eta)\,\mathrm{d}t$$

$$+ C_1\, C_{T9}\, (c_7 + 4C_1\, C_{T9}\, T)\sum_{j=1}^{m} \tau_j\, \|\eta_{j-1}^-\|^2_{\Omega_{t_{j-1}}}\Big) + 2\|\eta_m^-\|^2_{\Omega_{t_m}}$$

$$+ \frac{1}{2}\sum_{j=1}^{m} \|\{\eta\}_{j-1}\|^2_{\Omega_{t_{j-1}}} + \beta_0 \sum_{j=1}^{m}\int_{I_j} \|\eta\|^2_{DG,t}\,\mathrm{d}t, \quad m = 1, \dots, M,$$

*where $C^*$ is a constant independent of $\tau$.*

# Error estimate in terms of $h$ and $\tau$

- difficult open problem how to estimate the expression $\|\{\eta\}_{j-1}\|_{\Omega_{t_{j-1}}}$
- we omit the expression $\frac{1}{4}\sum_{j=1}^{m}\|\{e\}_{j-1}\|^2_{\Omega_{t_{j-1}}}$ and get

$$
\begin{aligned}
&\|e_m^-\|^2_{\Omega_{t_m}} + \frac{\beta_0}{2}\sum_{j=1}^{m}\int_{I_j}\|e\|^2_{DG,t}\,\mathrm{d}t \\
&\leq C_{AE}\Big(\sum_{j=1}^{m}\int_{I_j} C^* K_{t,j}(\eta)\,\mathrm{d}t + C_1\, C_{T9}\,(c_7 + 4C_1\, C_{T9}\, T)\sum_{j=1}^{m}\tau_j\,\|\eta_{j-1}^-\|^2_{\Omega_{t_{j-1}}}\Big) \\
&\quad +2\|\eta_m^-\|^2_{\Omega_{t_m}} + \beta_0\sum_{j=1}^{m}\int_{I_j}\|\eta\|^2_{DG,t}\,\mathrm{d}t, \quad m = 1,\dots,M
\end{aligned}
$$

# Error estimate in terms of $h$ and $\tau$

- the error can be written in the form

$$e(x,t) \quad = \quad \underbrace{(U(x,t) - \pi u(x,t))}_{\xi(x,t)} + \underbrace{(\pi u(x,t) - u(x,t))}_{\eta(x,t)}.$$

- terms $\eta(x,t)$ and $\xi(x,t)$ can be transferred to the reference domain using the ALE-mapping $\mathcal{A}_{h,t}^{m-1}$:

$$\tilde{\eta}(X,t) = \eta(\mathcal{A}_{h,t}^{m-1}(X),t), \quad \tilde{\xi}(X,t) = \xi(\mathcal{A}_{h,t}^{m-1}(X),t),$$

$$x = \mathcal{A}_{h,t}^{m-1}(X),\, X \in \Omega_{t_{m-1}},\, x \in \Omega_t,\; t \in [t_{m-1}, t_m].$$

- estimates for $\sum_{j=1}^m \int_{I_j} C^* K_{t,j}(\eta)\,\mathrm{d}t$, $\sum_{j=1}^m \tau_j \, \|\eta_{j-1}^-\|^2_{\Omega_{t_{j-1}}}$, $\|\eta_m^-\|^2_{\Omega_{t_m}}$, $\sum_{j=1}^m \int_{I_j} \|\eta\|^2_{DG,t}\,\mathrm{d}t$ using results for similar terms with $\tilde{\eta}$ and the substitution theorem

# Error estimate in terms of $h$ and $\tau$

## Theorem

*There exists a constant $C_{T12} > 0$ such that*

$$\|e_m^-\|^2_{\Omega_{t_m}} + \frac{\beta_0}{2}\sum_{j=1}^{m}\int_{I_j}\|e\|^2_{DG,t}\,\mathrm{d}t$$

$$\leq C_{T12}\Bigg(\sum_{j=1}^{m}\Bigg(h^{2(\mu-1)}|\tilde{u}|^2_{L^2(I_j;H^\mu(\Omega_{t_{j-1}}))} + h^{2(\mu-1)}\,\tau^2\left|\frac{\partial\tilde{u}}{\partial t}\right|^2_{L^2(I_j;H^\mu(\Omega_{t_{j-1}}))}$$

$$+\tau^{2(q+\gamma)}\big(|\tilde{u}|^2_{H^{q+1}(I_j;L^2(\Omega_{t_{j-1}}))} + |\tilde{u}|^2_{H^{q+1}(I_j;H^1(\Omega_{t_{j-1}}))}\big) + \tau^{2(q+1)}\left|\frac{\partial\tilde{u}}{\partial t}\right|^2_{H^q(I_j;H^1(\Omega_{t_j}}$$

$$+h^{2\mu}\sum_{j=2}^{m}\tau|\tilde{u}(t_{j-2}-)|^2_{H^\mu(\Omega_{t_{j-2}})} + h^{2\mu}|\tilde{u}(t_{m-1}-)|^2_{H^\mu(\Omega_{t_{m-1}})}\Bigg).$$

# ALE formulation of the compressible Navier-Stokes equations

- solution of the compressible Navier-Stokes system is a state vector $\boldsymbol{w} : \Omega_t \times [0, T] \to \mathbb{R}^4$

$$\frac{D\boldsymbol{w}}{Dt} + \sum_{s=1}^{2} \frac{\partial \boldsymbol{g}_s(\boldsymbol{w})}{\partial x_s} + \boldsymbol{w}\,\mathrm{div}\,\boldsymbol{z} = \sum_{s=1}^{2} \frac{\partial \boldsymbol{R}_s(\boldsymbol{w}, \nabla \boldsymbol{w})}{\partial x_s}.$$

- $\boldsymbol{w} = (\rho, \rho v_1, \rho v_2, E)^T \in \mathbb{R}^4$ - state vector
- $\boldsymbol{g}_s(\boldsymbol{w}) = \boldsymbol{f}_s(\boldsymbol{w}) - \boldsymbol{z}\boldsymbol{w}$, $s = 1, 2$ - modified inviscid fluxes
  $\boldsymbol{f}_s(\boldsymbol{w}) = (\rho v_s, \rho v_1 v_s + \delta_{1s} p, \rho v_2 v_s + \delta_{2s} p, (E + p) v_s)^T$
- $\boldsymbol{R}_s(\boldsymbol{w}, \nabla \boldsymbol{w})$ - viscous terms
  $\boldsymbol{R}_s(\boldsymbol{w}, \nabla \boldsymbol{w}) = (0, \tau_{s1}^V, \tau_{s2}^V, \tau_{s1}^V v_1 + \tau_{s2}^V v_2 + k \frac{\partial \theta}{\partial x_s})^T$
- completed by initial and boundary conditions

# Dynamic elasticity system

- elastic body is represented by a bounded polygonal domain $\Omega^b \subset \mathbb{R}^2$
- $\partial\Omega^b = \Gamma_D^b \cup \Gamma_N^b$, where $\Gamma_D^b \cap \Gamma_N^b = \emptyset$.
- Find a displacement function $\boldsymbol{u} : \overline{\Omega^b} \times [0, T] \to \mathbb{R}^2$ such that

$$
\begin{aligned}
\rho^b \frac{\partial^2 \boldsymbol{u}}{\partial t^2} + C_M^b \rho^b \frac{\partial \boldsymbol{u}}{\partial t} - \mathrm{div} \boldsymbol{P}(\boldsymbol{F}) &= \boldsymbol{f} && \text{in } \Omega^b \times [0, T], \\
\boldsymbol{u} &= \boldsymbol{u}_D && \text{in } \Gamma_D^b \times [0, T], \\
\boldsymbol{P}(\boldsymbol{F})\, \boldsymbol{n} &= \boldsymbol{g}_N && \text{in } \Gamma_N^b \times [0, T], \\
\boldsymbol{u}(\cdot, 0) = \boldsymbol{u}_0, \quad \frac{\partial \boldsymbol{u}}{\partial t}(\cdot, 0) &= \boldsymbol{z}_0 && \text{in } \Omega^b,
\end{aligned}
$$

# Type of material

Deformation mapping, deformation gradient, Jacobian:

$$\psi(\boldsymbol{X}, t) = \boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X}, t), \quad \boldsymbol{F} = \nabla\psi, \quad J = \det \boldsymbol{F} > 0$$

**Linear elasticity:**

$$\boldsymbol{P}(\boldsymbol{F}) := \lambda^b \operatorname{tr}(\boldsymbol{e}(\boldsymbol{u}))\mathbb{I} + 2\mu^b \boldsymbol{e}(\boldsymbol{u}), \quad \boldsymbol{e} = (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)/2$$

**Nonlinear elasticity:**

- St. Venant-Kirchhoff material

$$\boldsymbol{P}(\boldsymbol{F}) = \boldsymbol{F}(\lambda^b \mathrm{tr}(\boldsymbol{E})\mathbb{I} + 2\mu^b \boldsymbol{E}), \quad \boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}\right)$$

- Neo-Hookean material

$$\boldsymbol{P}(\boldsymbol{F}) = \mu^b(\boldsymbol{F} - \boldsymbol{F}^{-T}) + \lambda^b \log(\det \boldsymbol{F})\boldsymbol{F}^{-T}$$

# Discretization

**Fluid flow** and **elasticity problem**

- both discretized by the STDGM
- piecewise polynomial functions in space and time on the triangulation
- arbitrary polynomial degree

# Determination of the ALE mapping

## Artificial stationary linear elasticity problem

We seek $\boldsymbol{d} = (d_1, d_2)$ defined in $\Omega_{ref}$ as a solution of the elastic static system

$$\sum_{j=1}^{2} \frac{\partial \tau_{ij}^{a}(\boldsymbol{d})}{\partial X_j} = 0 \text{ in } \Omega_{ref}, \quad i = 1, 2,$$

where $\tau_{ij}^{a}$ are the components of the artificial stress tensor $\tau_{ij}^{a} = \delta_{ij}\lambda^{a}\mathrm{div}\boldsymbol{d} + 2\mu^{a} e_{ij}^{a}(\boldsymbol{d}),\ e_{ij}^{a}(\boldsymbol{d}) = \frac{1}{2}\left(\frac{\partial d_i}{\partial X_j} + \frac{\partial d_j}{\partial X_i}\right),\ i = 1, 2$. The Lamé coefficients $\lambda^{a}$ and $\mu^{a}$ are related to the artificial Young modulus $E^{a}$ and the artificial Poisson number $\nu^{a}$.

We get ALE mapping of $\overline{\Omega}_{ref}$ onto $\overline{\Omega}_t$ in the form

$$\mathcal{A}_t(\boldsymbol{X}) = \boldsymbol{X} + \boldsymbol{d}(\boldsymbol{X}, t), \quad \boldsymbol{X} \in \overline{\Omega}_{ref}, \tag{8}$$

for each time instant $t$.

# FSI problem

Coupling of the discrete flow problem and the structural problem is realized via the transmission conditions representing the continuity of the velocity and normal stress on the common boundary between fluid and structure.

## Transmission conditions

a) For linear elasticity:

$$\boldsymbol{P}(\boldsymbol{F}(\boldsymbol{X},t))\boldsymbol{n}(\boldsymbol{X}) = \boldsymbol{\tau}^f(\boldsymbol{x},t)\boldsymbol{n}(\boldsymbol{X}), \quad \boldsymbol{v}(\boldsymbol{x},t) = \frac{\partial \boldsymbol{u}(\boldsymbol{X},t)}{\partial t}$$

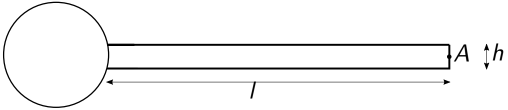
b) For nonlinear elasticity:

$$\boldsymbol{P}(\boldsymbol{F}(\boldsymbol{X},t))\boldsymbol{n}(\boldsymbol{X}) = \boldsymbol{\tau}^f(\boldsymbol{x},t)\mathrm{Cof}(\boldsymbol{F}(\boldsymbol{X},t))\boldsymbol{n}(\boldsymbol{X}), \quad \boldsymbol{v}(\boldsymbol{x},t) = \frac{\partial \boldsymbol{u}(\boldsymbol{X},t)}{\partial t}$$

where $\boldsymbol{x} = \boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X},t)$, $\boldsymbol{v}$ is the flow velocity, $\boldsymbol{\tau}^f = \{\tau_{ij}^f\}_{i,j=1}^2 = \{-p\delta_{ij} + \tau_{ij}^V\}_{i,j=1}^2$ represents the aerodynamical stress tensor.

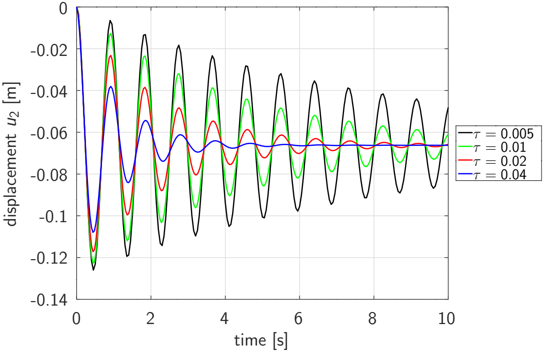
# Nonlinear elasticity benchmark problem

- **Turek-Hron benchmark problem**: elastic beam attached to a rigid cylinder ($l = 0.35$ m, $h = 0.02$ m)



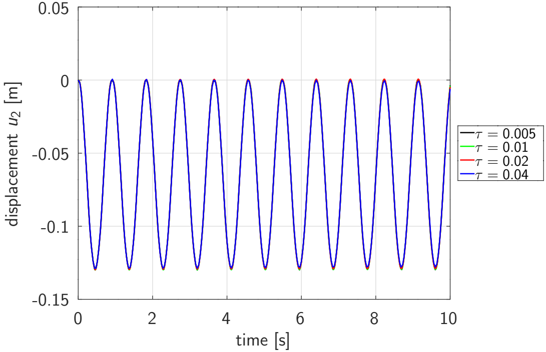

- evaluate the **displacement of the control point** $A = A(t)$ represented by the mean value, amplitude and frequency
- acting body force density
  $\boldsymbol{f} = \rho^b \boldsymbol{b}, \quad \text{where} \quad \boldsymbol{b} = (0, -2)^T \ [\mathrm{m\,s^{-2}}], \quad \rho^b = 1000 \ [\mathrm{kg\,m^{-3}}].$
- Young's modulus $1.4 \cdot 10^6$ and Poisson ratio 0.4
- $\boldsymbol{u}_D = \boldsymbol{0}$, $\boldsymbol{g}_N = \boldsymbol{0}$, St. Venant-Kirchhoff material

# Displacement of the point $A$ for STDGM $s = 1,\ q^* = 0$



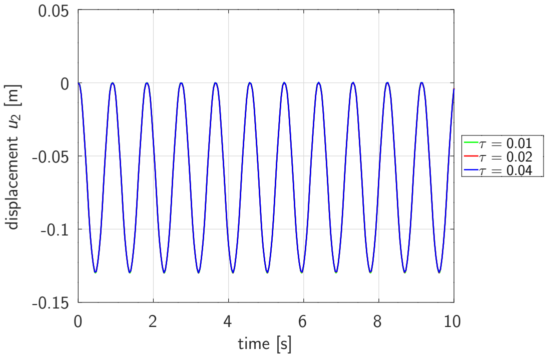

| method | $\tau$ | $u_1$ $\left[\times 10^{-3}\right]$ | | $u_2$ $\left[\times 10^{-3}\right]$ | |
|---|---|---|---|---|---|
| ref | | $-14.305 \pm 14.305$ | [1.0995] | $-63.607 \pm 65.160$ | [1.0995] |
| STDGM | 0.04 | $-7.203 \pm 0.002$ | [1.0712] | $-66.214 \pm 0.011$ | [1.0725] |
| STDGM | 0.02 | $-7.186 \pm 0.175$ | [1.0800] | $-66.130 \pm 0.789$ | [1.0775] |
| STDGM | 0.01 | $-7.200 \pm 1.564$ | [1.0887] | $-65.705 \pm 7.079$ | [1.0862] |
| STDGM | 0.005 | $-7.840 \pm 4.708$ | [1.0920] | $-65.409 \pm 21.393$ | [1.0900] |

# Displacement of the point $A$ for STDGM $s = 1,\ q^* = 1$



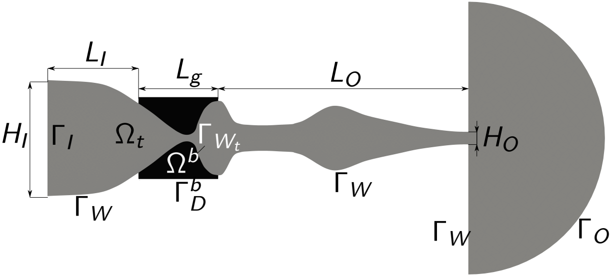

| method | $\tau$ | $u_1\ \left[\times 10^{-3}\right]$ | | $u_2\ \left[\times 10^{-3}\right]$ | |
|---|---|---|---|---|---|
| ref | | $-14.305 \pm 14.305$ | [1.0995] | $-63.607 \pm 65.160$ | [1.0995] |
| STDGM | 0.04 | $-14.072 \pm 14.043$ | [1.0925] | $-66.374 \pm 61.499$ | [1.0925] |
| STDGM | 0.02 | $-14.337 \pm 14.316$ | [1.0925] | $-66.456 \pm 62.556$ | [1.0925] |
| STDGM | 0.01 | $-14.546 \pm 14.526$ | [1.0950] | $-66.580 \pm 62.994$ | [1.0950] |
| STDGM | 0.005 | $-14.628 \pm 14.608$ | [1.0930] | $-66.623 \pm 63.153$ | [1.0930] |

# Displacement of the point $A$ for STDGM $s = 1, \ q^* = 2$




| method | $\tau$ | $u_1$ $[\times 10^{-3}]$ | | $u_2$ $[\times 10^{-3}]$ | |
|---|---|---|---|---|---|
| ref | | $-14.305 \pm 14.305$ | [1.0995] | $-63.607 \pm 65.160$ | [1.0995] |
| STDGM | 0.04 | $-14.497 \pm 14.497$ | [1.0925] | $-64.743 \pm 64.748$ | [1.0925] |
| STDGM | 0.02 | $-14.627 \pm 14.627$ | [1.0925] | $-65.088 \pm 64.711$ | [1.0925] |
| STDGM | 0.01 | $-14.672 \pm 14.672$ | [1.0950] | $-64.879 \pm 65.025$ | [1.0900] |

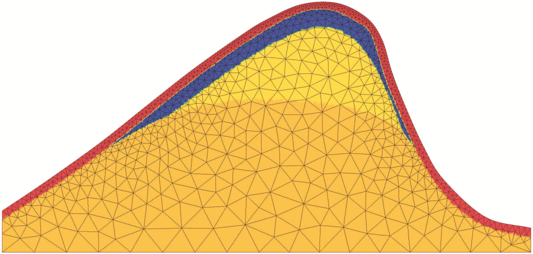
# Flow induced vocal folds vibrations




Computational domain: $L_I = 20.0$ mm, $L_g = 17.5$ mm, $L_O = 55.0$ mm, $H_I = 25.5$ mm, $H_O = 2.76$ mm. The radius of the semicircle is 3.0 cm.

- $v_{in} = 4$ m s$^{-1}$, $\mu = 1.80 \cdot 10^{-5}$ kg m$^{-1}$ s$^{-1}$, $\rho_{in} = 1.225$ kg m$^{-3}$, $p_{out} = 97611$ Pa,
- $Re = \rho_{in} v_{in} H_I / \mu = 6941.7$, $\kappa = 2.428 \cdot 10^{-2}$ kg m s$^{-3}$ K$^{-1}$, $c_v = 721.428$ m$^2$ s$^{-2}$ K$^{-1}$, $\gamma = 1.4$

# Flow induced vocal folds vibrations



- vocal folds are isotropic with constant material density $\rho^b = 1040$ kg m$^{-3}$
- the domain is divided into 4 regions with different material characteristics
- **fluid solver**: STDGM of degree 2 in space and degree 1 in time, $\tau = 1.0 \cdot 10^{-6}$ s, 17652 elements
- **elasticity solver**: STDGM of degree 1 in space and degree 1 in time, 5118 elements

# Flow induced vocal folds vibrations

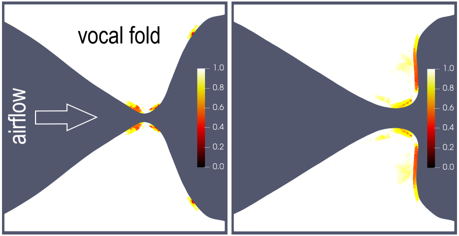
- compare the linear strain tensor $\boldsymbol{e}$ and the nonlinear Green strain tensor $\boldsymbol{E}$
- in case of the linear elasticity the stress tensor depends on the strain tensor $\boldsymbol{e} = (e_{ij})_{i,j=1}^{2}$
- in the case of nonlinear elasticity it depends on $\boldsymbol{E} = \boldsymbol{e} + \boldsymbol{E}^*$, where $\boldsymbol{E}^* = (E_{ij}^*)_{i,j=1}^{2}$.
- influence of the nonlinear part of the strain tensor is given by the ratio

$$R := \frac{\|\boldsymbol{e}\|}{\|\boldsymbol{E}\|} = \frac{\|\boldsymbol{e}\|}{\|\boldsymbol{e} + \boldsymbol{E}^*\|}.$$

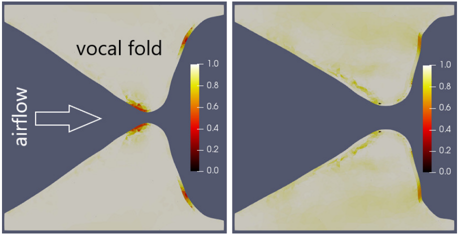
- if $R \approx 1$, then the **nonlinear part** of the strain tensor **has no influence** to the computation - the linear elasticity model is sufficient
- if $R \approx 0$, then the **nonlinear part strongly takes effect** - it is necessary to use a nonlinear elasticity model

# Comparing the linear and nonlinear elasticity model

- linear vs. neo-Hookean nonlinear model:




- linear vs. St. Venant-Kirchhoff nonlinear model:



# Conclusion

**Summary**

- discretization of the nonlinear convection-diffusion problem in a time-dependent domain by the ALE-STDGM
- generalization of the concept of discrete characteristic function in time-dependent domains
- proof of the unconditional stability of the method
- derivation of error estimates
- applications of the STDGM showing the accuracy and robustness of the proposed method

**Thank you for your attention!**